Multigrid for the SPIRAL prototype in Scala

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Multigrid Methods in SPIRAL (1)

- Recently: Generation of multigrid solvers with SPIRAL
- Paper: "Algebraic Description and Automatic Generation of Multigrid Methods in SPIRAL" (Bolten, Franchetti et al. 2017)
- Considered PDE: Discretized square 2D Poisson equation with Dirichlet boundary conditions

$$
\begin{aligned}
-\Delta u(x) &= f(x), \quad x \in \Omega := [0,1]^2, \\
u(x) &= 0, \quad x \in \partial\Omega
\end{aligned}
$$

- Specification of novel rewrite rules in the declarative language SPL

Multigrid Methods in SPIRAL (2)

$$
\begin{aligned}
\mathrm{MGSolvePDE}_{n,\omega,r,m} &\rightarrow [\mathrm{I}_{n^2} \mid 0_{n^2}] \cdot \left(\prod_{i=0}^{m-1} \mathrm{MGCycle}_{n,\omega,r} \right) \cdot \begin{bmatrix} 0_{n^2} \\ \mathrm{I}_{n^2} \end{bmatrix} \\
\mathrm{MGCycle}_{n,\omega,r} &\rightarrow \mathrm{CGC}_{n,\omega,r} \cdot \mathrm{Richardson}_{n,\omega,r} \\
\mathrm{CGC}_{n,\omega,r} &\rightarrow \begin{bmatrix} \mathrm{CoarseError}_{n,\omega,r} \\ 0_{n^2} \mid \mathrm{I}_{n^2} \end{bmatrix} \\
\mathrm{CoarseError}_{n,\omega,r} &\rightarrow \mathrm{Interpolate}_n \cdot \mathrm{Scatter}_n \cdot \mathrm{Solve}_{n,\omega,r} \cdot \mathrm{Gather}_n \cdot \mathrm{Residual}_n \\
\mathrm{Interpolate}_n &\rightarrow \mathrm{Tridiag}_n(\sqrt{2}/2, \sqrt{2}, \sqrt{2}/2) \otimes \mathrm{Tridiag}_n(\sqrt{2}/2, \sqrt{2}, \sqrt{2}/2) \\
\mathrm{Scatter}_n &\rightarrow \mathrm{S}_{1,2}^{n \times k} \otimes \mathrm{S}_{1,2}^{n \times k} \\
\mathrm{Solve}_{n,\omega,r} &\rightarrow \begin{cases} \frac{1}{4}\,\mathrm{I}_1, & n = 3 \\ [\mathrm{I}_{k^2} \mid 0_{k^2}] \cdot \mathrm{MGCycle}_{k,\omega,r} \cdot \begin{bmatrix} 0_{k^2} \\ \mathrm{I}_{k^2} \end{bmatrix}, & n > 3 \end{cases} \\
\mathrm{Gather}_n &\rightarrow \mathrm{G}_{1,2}^{k \times n} \otimes \mathrm{G}_{1,2}^{k \times n} \\
\mathrm{Residual}_n &\rightarrow [\mathrm{Tridiag}_n(1,-2,1) \otimes \mathrm{I}_n + \mathrm{I}_n \otimes \mathrm{Tridiag}_n(1,-2,1) \mid \mathrm{I}_{n^2}] \\
\mathrm{Richardson}_{n,\omega,r} &\rightarrow \prod_{i=0}^{r-1} \begin{bmatrix} \mathrm{ResidueLaplace}_{n,\omega} & \omega\,\mathrm{I}_{n^2} \\ 0_{n^2} & \mathrm{I}_{n^2} \end{bmatrix} \\
\mathrm{ResidueLaplace}_{n,\omega} &\rightarrow \mathrm{Tridiag}_n(\omega, 0.5 - 2\omega, \omega) \otimes \mathrm{I}_n + \mathrm{I}_n \otimes \mathrm{Tridiag}_n(\omega, 0.5 - 2\omega, \omega)
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where $k = (n-1)/2$

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SpiralS

- Case study: partial reimplementation of SPIRAL in Scala
  → only covering the DFT
- Paper: “SPIRAL in Scala: Towards the Systematic Construction of Generators for Performance Libraries” (Püschel, Odersky et al. 2013)
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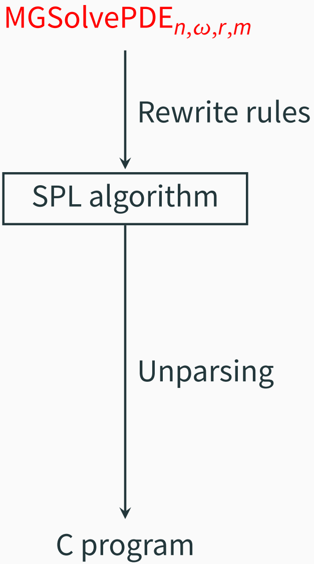
My Work

Is it possible to achieve an extension of SpiralS to the domain of multigrid methods similarly to SPIRAL?

Generator Architecture



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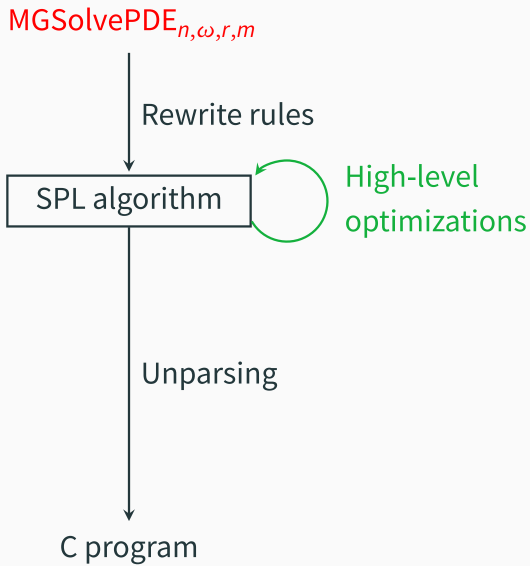

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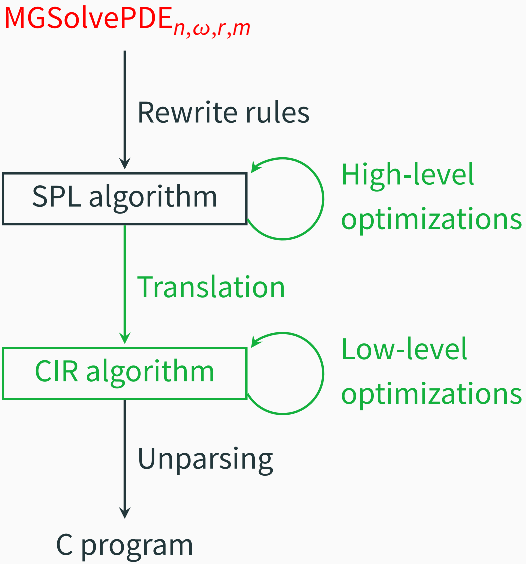

Translating from SPL to CIR (1)

$y = \text{Tridiag}_n(a, b, c) \cdot x$

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$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} b \cdot x_0 + c \cdot x_1 \\ a \cdot x_0 + b \cdot x_1 + c \cdot x_2 \\ \vdots \\ a \cdot x_{n-3} + b \cdot x_{n-2} + c \cdot x_{n-1} \\ a \cdot x_{n-2} + b \cdot x_{n-1} \end{bmatrix} = \begin{bmatrix} b & c & & & \\ a & b & c & & \\ & \ddots & \ddots & \ddots & \\ & & a & b & c \\ & & & a & b \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{bmatrix}$$

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```
case Const(Tridiag(n, a, b, c)) ⇒
  y(0) = b * x(0) + c * x(1)
  for (i ← 1 until n - 1)
    y(i) = a * x(i - 1) + b * x(i) + c * x(i + 1)
  y(n - 1) = a * x(n - 2) + b * x(n - 1)
```

Translating from SPL to CIR (2)

$y = (A + B) \cdot x$

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$$y = (A + B) \cdot x = \underbrace{(A \cdot x)}_{=:\, a} + \underbrace{(B \cdot x)}_{=:\, b} = a + b$$

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```
case Def(Plus(lhs, rhs, rows, _)) ⇒
  val B = translate(rhs)
  val A = translate(lhs)
  val b = NewArray[Double](rows)
  val a = NewArray[Double](rows)
  B(b, x)
  A(a, x)
  for (i ← 0 until rows) y(i) = a(i) + b(i)
```

Translating from SPL to CIR (2)

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```

Stencil Computations (1)

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \text{Tridiag}_3(a, b, c) \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b & c & 0 \\ a & b & c \\ 0 & a & b \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

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$$y_0 = \begin{bmatrix} 0 & a & b & c & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

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$$y_1 = \begin{bmatrix} 0 & a & b & c & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_0 \\ x_1 \\ x_2 \\ 0 \end{bmatrix}$$

Stencil Computations (1)

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$$y_2 = \begin{bmatrix} 0 & a & b & c & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix}$$

Stencil Computations (2)

- Tridiagonal matrices characterized by coefficient vectors:

$$T := \text{Tridiag}_n(a, b, c) \qquad U := \text{Tridiag}_n(\alpha, \beta, \gamma) \qquad T + U$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \qquad \begin{bmatrix} a + \alpha \\ b + \beta \\ c + \gamma \end{bmatrix}$$

Stencil Computations (2)

- Tridiagonal matrices characterized by coefficient vectors:

$$T \coloneqq \mathrm{Tridiag}_n(a,b,c) \qquad U \coloneqq \mathrm{Tridiag}_n(\alpha,\beta,\gamma) \qquad T+U$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \qquad \begin{bmatrix} a+\alpha \\ b+\beta \\ c+\gamma \end{bmatrix}$$

- Observation: let $T \coloneqq \mathrm{Tridiag}_n(a,b,c)$, then

$$T \otimes T = \begin{bmatrix} bT & cT & & & \\ aT & bT & cT & & \\ & \ddots & \ddots & \ddots & \\ & & aT & bT & cT \\ & & & aT & bT \end{bmatrix}, \qquad I_n \otimes T = \begin{bmatrix} T & 0 & & & \\ 0 & T & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & T & 0 \\ & & & 0 & T \end{bmatrix}$$

$\rightarrow$ harness the structure of those matrices using stencils

Stencil Computations (3)

- Coefficient matrix for $T \otimes U$:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix} = \begin{bmatrix} a\alpha & a\beta & a\gamma \\ b\alpha & b\beta & b\gamma \\ c\alpha & c\beta & c\gamma \end{bmatrix}$$

- To add Kronecker products of tridiagonal matrices $T_0, \ldots, T_{n-1}$

$$(T_0 \otimes T_1) + (T_2 \otimes T_3) + \ldots + (T_{n-2} \otimes T_{n-1})$$

$\rightarrow$ Add their coefficient matrices

- Generalizes to Toeplitz matrices

Summary

- Fully-fledged, albeit very simple and highly specialized code generator
- Encouraging results for simple SPL expressions
- SPIRAL-generated code for MGCycle is much shorter

Thank you for your attention!

Questions?